Soluble Groups and p-Groups

Talk 4: Finite p-groups

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p-groups

G is a (finite) p-group

$$\Leftrightarrow$$

$$|G| = p^n$$

(p prime, n natural number)

Classification

Central aim in group-theory:

Classify p-groups up to isomorphism.

First steps

Easy hand computations show:

order	number of groups	comments
p	1	C_p (cyclic)
p^2	2	C_p^2, C_{p^2}
p^3	5	3 abelian, 2 non-abelian
p^4	15 (or 14 if $p = 2$)	

Further steps I

More difficult: There are

$$2p + 61 + (4, p-1) + 2(3, p-1)$$

groups of order p^5 if $p > 3$.

(Master thesis by Girnat (2003) is basis for SmallGroups version)

Further steps II

Still more difficult: There are

$$3p^2 + 39p + 344 + 24(3, p-1) + 11(4, p-1) + 2(5, p-1)$$

groups of order p^6 if $p > 3$.

(Newman, O'Brien and Vaughan-Lee (2003))

Further steps III

Still more difficult: There are

$$3p^5 + \dots$$

groups of order p^7 if $p > 5$.

(O'Brien and Vaughan-Lee (2004) - over 700 pages of proof)

Higman's Porc Conjecture

Higman conjectured that the number $f(n)$ of groups of order p^n is a polynomial in residue classes (porc) if p is large enough.

(Confirmed for $n \leq 7$ by above results.)

p-group generation

Algorithm to compute the groups of order p^n:

- p-group generation: Newman & O'Brien (1990)
- Variation to enumerate: Eick & O'Brien (2000)
- Use the structure of the groups.

Groups of order 2^n

Order	Number	Comment
2^1	1	
2^2	2	
2^3	5	
2^4	14	Hölder 1893
2^5	51	Miller 1898
2^6	267	Hall & Senior 1964
2^7	2328	James, Newman & O'Brien 1990
2^8	56 092	O'Brien 1991
2^9	10 494 213	Eick & O'Brien 2000
2^{10}	49 487 365 422	Eick & O'Brien 2000

A new invariant: the coclass

Let G be a group of order p^n.

- Let $G_1 = G$ and $G_{i+1} = [G_i, G]$ for $i \geq 1$.
- Obtain $G = G_1 > \ldots > G_c > G_{c+1} = \{1\}$ the lower central series of G.
- The coclass $cc(G)$ of G is defined as

$$cc(G) = n - c.$$

A new invariant: the coclass

Aim: Classify p-groups by coclass

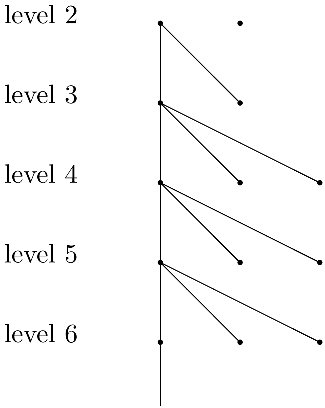
(Suggested by Leedham-Green & Newman 1980)

The coclass graph

The graph $\mathcal{G}(p, r)$:

- Vertices (on level n):
  $\leftrightarrow$ groups of order p^n and coclass r (up to isomorphism).
- G and H are connected by an edge:
  $\leftrightarrow$ there exists $N \trianglelefteq H$ with $|N| = p$ and $H/N \cong G$.

Example: $\mathcal{G}(2, 1)$



The structure of $\mathcal{G}(p, r)$

Not difficult to observe:

- $\mathcal{G}(p, r)$ has infinitely many vertices and edges.
- $\mathcal{G}(p, r)$ is a forest.

Infinite paths and trees

Consider infinite paths in more detail:

- Let $G = G_0, G_1, \ldots$ be an infinite path in $\mathcal{G}(p, r)$.
- Let $\mathcal{T}_G$ be the full tree of descendants of G in $\mathcal{G}(p, r)$.
- Then $\mathcal{T}_G$ is a coclass tree if $\mathcal{T}_G$ contains exactly one infinite path.

Infinite paths and pro-p-groups

- Let $G = G_0, G_1, \ldots$ be an infinite path in $\mathcal{G}(p, r)$.
- Then $P = lim G_i$ is an infinite pro-p-group of coclass r.

The structure of $\mathcal{G}(p, r)$

The proof of the coclass conjectures (Leedham-Green (1994), Shalev (1994)) implies that:

- There are only finitely many infinite pro-p-groups of coclass r.
- Every infinite pro-p-group of coclass r corresponds to a coclass tree in $\mathcal{G}(p, r)$.
- Yields that $\mathcal{G}(p, r)$ consists of finitely many coclass trees (and finitely many other groups).

Asymptotic estimates for number of coclass trees

An asymptotic estimate on the number $f(p,r)$ of infinite pro-p-groups of coclass r (Eick 2006):

$$p^{p^{r-1}} \leq f(p,r) \leq r^2 p^{r^3 p^{2(r-1)}}.$$

Branches

Let $\mathcal{T}_G$ be a coclass tree with infinite path $G = G_0, G_1, \ldots$.

- Let $\mathcal{B}_i$ be the subtree of $\mathcal{T}_G$ of all descendants of G_i which are not descendants of G_{i+1}.
- Let $\mathcal{B}_{i,k}$ be the subtree of $\mathcal{B}_i$ of groups of distance at most k from the root G_i.
- Then $\mathcal{B}_{i,k}$ is a branch of $\mathcal{T}_G$.

Periodicity

Theorem: For every k there exist d and $l = l(k)$ so that $\mathcal{B}_{i,k} \cong \mathcal{B}_{i+d,k}$ for all $i \geq l$.

Conjectured by Newman & O'Brien (1999)
Proved by Du Sautoy using Zeta-Functions (2001)
Proved by Eick & Leedham-Green using Group-Theory (2006)

- Allows to define infinite coclass families.
- For $p = 2$ there exists $k = k(r)$ with $\mathcal{B}_i = \mathcal{B}_{i,k}$.

Algorithm I

Aim: classify the infinite pro-p-groups of coclass r.

Idea: use the structure of these groups.

Infinite pro-p-groups of finite coclass

The structure of an infinite pro-p-group P of coclass r:

- $Z_\infty(P)$ is a finite subgroup of P of order p^s, say, with $s < r$.
- $P/Z_\infty(P)$ has coclass $r - s$ and is a p-adic space group.
- Thus $P \geq T \geq Z_\infty(P)$ with P/T and $Z_\infty(P)$ finite and $T/Z_\infty(P) \cong \mathbb{Z}_p^d$.

Algorithm I

Algorithms: (Eick, Greve 2005+2007)

- Classify (rational) p-adic space groups of coclass r.
- Determine their extensions by finite groups.

In GAP

Determined the p-adic (rational) space groups of coclass r:

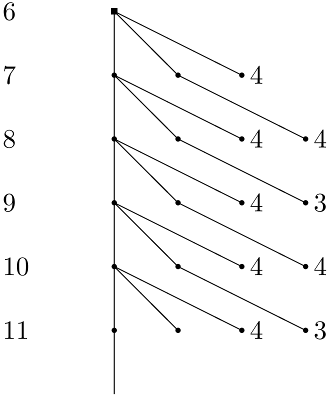
r	$p=2$	$p=3$	$p=5$
1	1	1	1
2	2	10	95
3	21	1271	1 110 136 753 555 665
4	268	137 299 952 383	
5	15013		

Algorithm II

Let P be the pro-p-group of a coclass tree $\mathcal{T}_G$

- Use GAP to construct finite parts of $\mathcal{T}_G$ from P.
- Yielded significant new insights into the structure of coclass trees.

Example: A tree in $\mathcal{G}(2,2)$



Application I

Theorem:

- The groups in an infinite coclass family can be described by a single parametrised presentation.
- The 2-groups of coclass r can be classified by finitely many presentations.

(Eick & Leedham-Green 2006)

Application II

Conjecture: Let G be a non-abelian p-group. Then $|G| \mid |Aut(G)|$.

Theorem: Among the 2-groups of coclass r there are at most finitely many counterexamples to this conjecture.

Reason: The automorphism groups of the groups in an infinite coclass family are periodic.

(Eick 2006)

Application III

An old problem: Find all p-groups with trivial Schur multiplicator.

Theorem: Let $p > 2$. Among the p-groups of coclass r there are at most finitely many with trivial Schur multiplicator.

Theorem: Among the 2-groups of coclass r there are infinite coclass families of groups with trivial Schur multiplicator.

Reason: Let $(G_x \mid x \in \mathbb{N})$ an infinite coclass family. Then $M(G_x)$ is of type $(p^{r_1 x + s_1}, \ldots, p^{r_l x + s_l})$.

(Eick 2007/Feichtenschlager 2010)