Finitely presented groups 1

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LMS Short Course on Computational Group Theory
29 July – 2 August 2013

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$\langle D, E, F \mid DEFED, EFD, D^2EF \rangle$ is the trivial group $\{1\}$.

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- $uv \sim uxx^{-1}v$ for all $u, v \in \hat{X}^*$ and all $x \in \hat{X}$, and
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Definition/Proposition (Finitely presented group)

The set $G := \hat{X}^*/\sim$ of equivalence classes is a group with concatenation as product. It is called the finitely presented group with generators X and relations R and is denoted by $G := \langle X \mid R \rangle$.

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e.g. in $G = \langle S, T \mid S^6, T^2, STST \rangle$:

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Theorem (Universal property (Dyck, 1882))

Let $G := \langle X \mid R \rangle$ be a finitely presented group.
Let H be any group, $f : X \to H$ any map and set $f(x^{-1}) := f(x)^{-1}$ for $x \in X$. If

$$f(x_1) \cdot f(x_2) \cdot \cdots \cdot f(x_k) = 1 \in H \qquad (*)$$

for all $r = x_1 x_2 \cdots x_k \in R$, then there is a unique group homomorphism $\tilde{f} : G \to H$ with $\tilde{f}(\overline{x}) = f(x)$ for all $x \in X$.

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$\left\langle C \mid C^k \right\rangle$ has all cyclic groups of order o dividing k as quotients.

In a seminal paper, Max Dehn formulated three fundamental problems:

Problem (Word problem (Dehn 1911))

For $G := \langle X \mid R \rangle$ and a word $w \in \hat{X}^*$, decide in finitely many steps whether or not $\overline{w} = 1 \in G$.

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For $G := \langle X \mid R \rangle$ and $H := \langle Y \mid S \rangle$, decide in finitely many steps whether or not $G \cong H$, or even: for a map $f : X \to \hat{Y}^*$, decide in finitely many steps whether or not f induces an isomorphism $\tilde{f} : G \to H$.

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We would like to have algorithms for these decision problems . . .

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$$w \sim \prod_{i=1}^{k} w_i^{-1} r_i w_i \qquad \text{in } F = \langle X \mid \rangle$$

for a $k \in \mathbb{N}$ and some $w_i \in \hat{X}^*$ and $r_i \in R \cup R^{-1}$.

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Definition (Dehn function)

$$\delta(\ell) := \max\{A(w) \mid w \in \hat{X}^* \text{ of length } \ell \text{ and } \overline{w} = 1\}$$

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Nevertheless, in GAP you can work with finitely presented groups:

`http://tinyurl.com/MNGAPsess/GAP_FP_1.g`

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We can then replace w everywhere by v.

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Rotating or inverting a relator does not change the group.

If $vw \in R$ with $v, w \in \hat{X}^*$, then $\overline{v} = \overline{w^{-1}} \in G$.

In other words: $vu \sim vww^{-1}u \sim w^{-1}u$,
so we can replace v by w^{-1} in any other relator.

In particular, if $v \in \hat{X}$ and w does not contain the letter v, we can replace v everywhere by w^{-1}.

We can then remove the relator vw and the generators v, v^{-1} from $\hat{X}$.

Conversely, if a word w shows up frequently in the relators, we can add generators v, v^{-1} to $\hat{X}$ and add the relator $v^{-1}w$.

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All these steps are called “Tietze transformations”, they change the presentation, but not the isomorphism type of the group.

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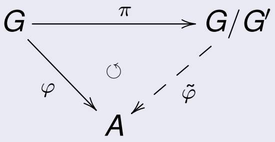
G/G' is the largest abelian quotient of G, we have:

Proposition

If $\varphi : G \to A$ is any group homomorphism to an abelian group A, then there is a unique group homomorphism $\tilde{\varphi} : G/G' \to A$ with $\varphi = \pi\tilde{\varphi}$:



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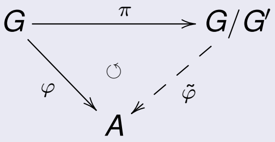

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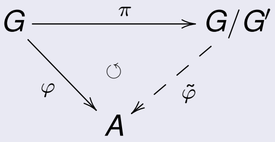

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We immediately get:

$$G/G' \cong H := \left\langle X \mid R, \{xyx^{-1}y^{-1} \mid x, y \in X\} \right\rangle$$

However, $H := \langle X \mid R, \{xyx^{-1}y^{-1} \mid x, y \in X\} \rangle$ is naturally a quotient of the free abelian group

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If $N := S \cdot M \cdot T$ is the Smith normal form of M (with S, T invertible), then $H \cong \mathbb{Z}^n/(\text{row space of } N)$ and we can read off the structure.

More concretely, if for example

$$N = \begin{bmatrix} 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & 2 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 6 & \cdot & \cdot \\ \cdot & \cdot & \cdot & 0 & \cdot \end{bmatrix} = S \cdot M \cdot T,$$

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GAP can do all this:

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