Finitely presented groups 2

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LMS Short Course on Computational Group Theory
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Let $G := \langle X \mid R \rangle$. There is a bijection:

$$\{H \leq G \mid [G : H] < \infty\} \xrightarrow{\cong} \{\varphi : G \to S_n \mid n \in \mathbb{N}, \varphi \text{ a grp. hom.}\}$$

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The conjugacy classes of finite index subgroups

are in bijection with

the equivalence classes of actions on finitely many points.

G is finitely generated, we describe finite actions by coset tables:

Example (A coset table)

Let $G := \left\langle c, d \mid c^2 = 1 = d^3 = (cd)^8 = [c, d]^4 \right\rangle$

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$$\begin{aligned} \text{Let } G &:= \left\langle c, d \mid c^2 = 1 = d^3 = (cd)^8 = [c,d]^4 \right\rangle \\ &\cong L_2(7):2 \cong \langle (2,4)(3,5)(6,8), (1,2,3)(5,6,7) \rangle \leq S_8. \end{aligned}$$

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Coset #	c	c^{-1}	d	d^{-1}
1	1	1	2	3
2	4	4	3	1
3	5	5	1	2
4	2	2	4	4
5	3	3	6	7
6	8	8	7	5
7	7	7	5	6
8	6	6	8	8

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5	3	3	6	7
6	8	8	7	5
7	7	7	5	6
8	6	6	8	8

Here, $H = \langle c, dcdc^{-1}d^{-1} \rangle$.

Todd-Coxeter

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A "name" of a coset is a number and a word representing the coset.

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A "name" of a coset is a number and a word representing the coset.

We make up new names and draw conclusions as we go and hope. . .

Let $G := \langle a, b \mid a^2, b^3, abab \rangle$ and $H := \langle ab \rangle$.

Events:

#	coset	a	a^{-1}	b	b^{-1}
1	H				

We start with an empty table like this.

Let $G := \langle a, b \mid a^2, b^3, abab \rangle$ and $H := \langle ab \rangle$.

#	coset	a	a^{-1}	b	b^{-1}
1	H	2			
2	Ha		1		

We call the coset Ha number 2, a definition.

Events:

Def. $2 := Ha$

Let $G := \langle a, b \mid a^2, b^3, abab \rangle$ and $H := \langle ab \rangle$.

#	coset	a	a^{-1}	b	b^{-1}
1	H	2	2		
2	Ha	1	1		

Events:

Def. $2 := Ha$
Ded. $Haa = H$

Note $Haa = H$, since $a^2 = 1$, this is a deduction.

Let $G := \langle a, b \mid a^2, b^3, abab \rangle$ and $H := \langle ab \rangle$.

#	coset	a	a^{-1}	b	b^{-1}
1	H	2	2		2
2	Ha	1	1	1	

Events:

Def. $2 := Ha$
Ded. $Haa = H$
Ded. $Hab = H$

Note $Hab = H$, since $ab \in H$, this is a deduction.

Let $G := \langle a, b \mid a^2, b^3, abab \rangle$ and $H := \langle ab \rangle$.

#	coset	a	a^{-1}	b	b^{-1}
1	H	2	2	3	2
2	Ha	1	1	1	
3	Hb	4			1
4	Hba		3		

Next we define $3 := Hb$ and $4 := Hba$.

Events:

Def. $2 := Ha$
Ded. $Haa = H$
Ded. $Hab = H$
Def. $3 := Hb$
Def. $4 := Hba$

Let $G := \langle a, b \mid a^2, b^3, abab \rangle$ and $H := \langle ab \rangle$.

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1	H	2	2	3	2
2	Ha	1	1	1	
3	Hb	4	4		1
4	Hba	3	3		

Deduce $Hbaa = Hb$.

Events:

Def. $2 := Ha$
Ded. $Haa = H$
Ded. $Hab = H$
Def. $3 := Hb$
Def. $4 := Hba$
Ded. $Hbaa = Hb$

Let $G := \langle a, b \mid a^2, b^3, abab \rangle$ and $H := \langle ab \rangle$.

#	coset	a	a^{-1}	b	b^{-1}
1	H	2	2	3	2
2	Ha	1	1	1	
3	Hb	4	4	5	1
4	Hba	3	3		
5	Hbb				3

Define $5 := Hbb$.

Events:

Def. $2 := Ha$
Ded. $Haa = H$
Ded. $Hab = H$
Def. $3 := Hb$
Def. $4 := Hba$
Ded. $Hbaa = Hb$
Def. $5 := Hbb$

Let $G := \langle a, b \mid a^2, b^3, abab \rangle$ and $H := \langle ab \rangle$.

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3	Hb	4	4	5	1
4	Hba	3	3		
5	Hbb			1	3

Events:

[...]
Ded. $Hab = H$
Def. $3 := Hb$
Def. $4 := Hba$
Ded. $Hbaa = Hb$
Def. $5 := Hbb$
Ded. $Hbbb = H$

Deduce $Hbbb = H$. Thus Hb^{-1} is both 2 and 5!

Let $G := \langle a, b \mid a^2, b^3, abab \rangle$ and $H := \langle ab \rangle$.

#	coset	a	a^{-1}	b	b^{-1}
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3	Hb	4	4	2	1
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5	Hbb	–	–	–	–

Events:

[...]
Def. $3 := Hb$
Def. $4 := Hba$
Ded. $Hbaa = Hb$
Def. $5 := Hbb$
Ded. $Hbbb = H$
Coi. $5 = 2$

Conclude $Ha = Hbb$, replace 5 by 2: a coincidence.

Let $G := \langle a, b \mid a^2, b^3, abab \rangle$ and $H := \langle ab \rangle$.

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Def. $4 := Hba$
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Note $Habab = H$, a deduction that is already known.

Let $G := \langle a, b \mid a^2, b^3, abab \rangle$ and $H := \langle ab \rangle$.

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2	Ha	1	1	1	3
3	Hb	4	4	2	1
4	Hba	3	3	2	
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Use $Ha \cdot abab = Ha$, deduce $Hbab = Ha$.

Events:

[. . .]
Def. $4 := Hba$
Ded. $Hbaa = Hb$
Def. $5 := Hbb$
Ded. $Hbbb = H$
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Events:

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Ded. $Hbaa = Hb$
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Table closed.

Conclude $Hba = Hb$, replace 4 by 3. Table is closed.

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Indeed, we have found a permutation representation on 3 points. The subgroup H fixes the first point.

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Since we have checked all relations we have found a group homomorphism from G to S_3.

`http://tinyurl.com/MNGAPsess/GAP_FP_4.g`

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However:

No strategy is always optimal.

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However:

No strategy is always optimal.

Runtime and memory usage vary enormously with the strategy.

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Theorem (Termination of the Todd-Coxeter procedure)

Assume $[G : H] < \infty$ and a deterministic strategy with:

- *all entries will eventually be filled,*
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Then the Todd-Coxeter procedure terminates eventually.

- No limit on memory and runtime is known a priori.
- A completed coset enumeration with $H = \{1\}$ proves G to be finite and determines the order.
- An unfinished coset enumeration proves nothing whatsoever.

Low index

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- use backtrack search,
- determine the point stabiliser H in each case, and
- remove equivalent actions (conjugate subgroups H).

$\Longrightarrow$ This very quickly becomes impractical for larger k.

Low index: an example

Let $G := \langle a, b \mid a^2, b^3, abab \rangle$ and $k = 3$.

#	a	b	b^{-1}
1			
2			
3			

Guesses:

We start with an empty table like this.

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Let $G := \langle a, b \mid a^2, b^3, abab \rangle$ and $k = 3$.

#	a	b	b^{-1}
1	1		
2			
3			

Guesses:

$1a = 1$

We first assume $1a = 1$. Nothing follows.

Low index: an example

Let $G := \langle a, b \mid a^2, b^3, abab \rangle$ and $k = 3$.

#	a	b	b^{-1}
1	1	2	
2			1
3			

Guesses:

$1a = 1$
$1b = 2$ (wlog)

$1b = 1$ would be intransitive, so (wlog) $1b = 2$.

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#	a	b	b^{-1}
1	1	2	2
2		1	1
3			

Guesses:

$1a = 1$
$1b = 2$ (wlog)
$2b = 1$

From $2b = 1$ would follow $1bbb = 2$, a contradiction.

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1	1	2	
2		3	1
3			2

Guesses:

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So we backtrack and conclude $2b = 3$.

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It follows that $3b = 1$ for $1bbb = 1$ to hold.

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3	3	1	2

Guesses:

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$1b = 2$ (wlog)

$2a = 2$ would imply $3a = 3$ and then $1abab = 3$.

Low index: an example

Let $G := \langle a, b \mid a^2, b^3, abab \rangle$ and $k = 3$.

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Guesses:

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Thus we have $2a = 3$ and everything is good.

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For this solution $H = \langle a \rangle$.

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Guesses:

$1a = 1$
$1b = 2$ (wlog)

Thus we have $2a = 3$ and everything is good.

For this solution $H = \langle a \rangle$.

To go on, we would change the assumption $1a = 1$ to $1a = 2$ (wlog) and continue the search.

`http://tinyurl.com/MNGAPsess/GAP_FP_5.g`

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