Finitely presented groups 3

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LMS Short Course on Computational Group Theory
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The following set generates E:

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Theorem (Nielsen-Schreier)

If T is prefix-closed, then E is a free group on Y.

Recall:

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$$w = (t_0 x_1 t_1^{-1})(t_1 x_2 t_2^{-1}) \cdots (t_{k-1} x_k t_k^{-1})$$

and all $t_{i-1} x_i t_i^{-1}$ are either 1_F or in Y.

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Assume from now on that T is prefix-closed.

Theorem (Reidemeister-Schreier)

For $G = F/\langle\langle R\rangle\rangle$, $H = E/\langle\langle R\rangle\rangle$, T and Y as above, if T is prefix-closed, then $H = E/N$ is isomorphic to

$$H' := \left\langle Y \mid \rho(twt^{-1}) \text{ for all } t \in T, w \in R\right\rangle.$$

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Example (The dihedral group of order 8)

Let $G := \langle s, t \mid s^4, t^2, stst \rangle$ and $H := \langle s^2, t \rangle$. We know that $|G| = 8$ and H is a Klein four group.

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Reidemeister-Schreier now gives the following relators:

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Thus, we get that $H \cong \langle A, B, C \mid B^2, A^2, C^2, CBA, BAC \rangle$.

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`http://tinyurl.com/MNGAPsess/GAP_FP_6.g`

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First idea: Abelian invariants, but what if they are all positive?

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If a RWS is complete, then every $\Leftrightarrow$ class contains exactly one irreducible element and all words in the class can be rewritten to it.

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How can it ever happen, that $a \to b$ and $a \to c$, but that b and c cannot be rewritten to any common word d?

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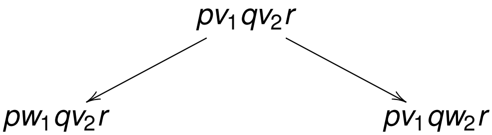

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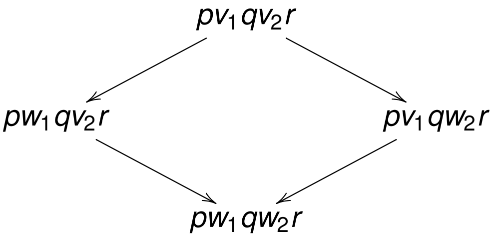

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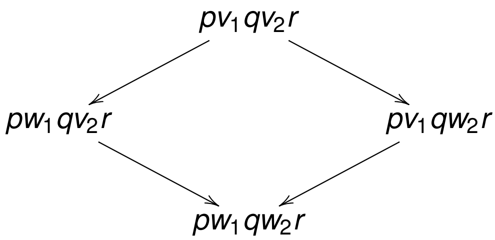

Thus: the left hand sides have to overlap!

Definition (Critical pair)

A pair of rules $v_1 \to w_1$ and $v_2 \to w_2$ is called a critical pair, if:

- $v_1 = rs$ and $v_2 = st$ for some $r, s, t \in A^*$, or
- $v_1 = rst$ and $v_2 = s$ for some $r, s, t \in A^*$,

with $s \neq \varepsilon$ in both cases.

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A RWS is locally confluent if and only if the following conditions are fulfilled for all critical pairs $v_1 \to w_1$ and $v_2 \to w_2$:

- *If $v_1 = rs$ and $v_2 = st$, then $\exists w \in A^*$ with $w_1 t \Rightarrow w$ and $rw_2 \Rightarrow w$.*
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We can check confluence of a finite, terminating RWS algorithmically!

Definition (Reduction ordering)

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Idea of the Knuth-Bendix completion procedure

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- rewrite $w_1 t \Rightarrow w_1'$ and $rw_2 \Rightarrow w_2'$ with w_1' and w_2' irreducible,
- if $w_1' \neq w_2'$, then
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If the RWS is already or becomes confluent this procedure terminates.

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Remark

With a complete RWS we have a good way to decide $v \Leftrightarrow w$.

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