Finitely presented groups 4

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LMS Short Course on Computational Group Theory
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Definition (Condition $C'(\lambda)$)

We say $\langle X \mid R \rangle$ is $C'(\lambda)$, if:

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($|r|$ is the length in letters).

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Definition (Condition $T(q)$)

We say $\langle X \mid R \rangle$ is $T(q)$, if the following holds:

- Let $3 \leq h < q$ and $(r_1, r_2, \ldots, r_h) \in R^h$ with no successive elements r_i, r_{i+1} or r_h, r_1 an inverse pair. Then at least one of the products $r_1 r_2, r_2 r_3, \ldots, r_h r_1$ is reduced without cancellation.

Theorem (Lyndon, Schupp)

Let $G = \langle X \mid R \rangle$ with R closed under rotation and inversion and all $r \in R$ are reduced. If $\langle X \mid R \rangle$ fulfills at least one of:

- *$C'(1/6)$ and $T(3)$, or*
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Definition (Dehn RWS)

Write all $r \in R$ as $r = ab$ with $|a| > |b|$ and define a rule $a \to b^{-1}$.

Algorithm (Dehn's algorithm)

Let $G = \langle X \mid R \rangle$ and let $\mathcal{R}$ be a length-reducing RWS for $\hat{X} = X \cup X^{-1}$.

1. **Input:** a word $w \in \hat{X}^*$.
2. Freely reduce w.
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- For a word of length n, this terminates after at most n steps.

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Lemma

If $G = \langle X \mid R \rangle$ is small cancellation, then Dehn works and every word $w \in \hat{X}^*$ of length n that is equal to 1 in G is the product of at most n conjugates of a relator. Thus, the Dehn function $\delta(n) \leq n$ for all n.

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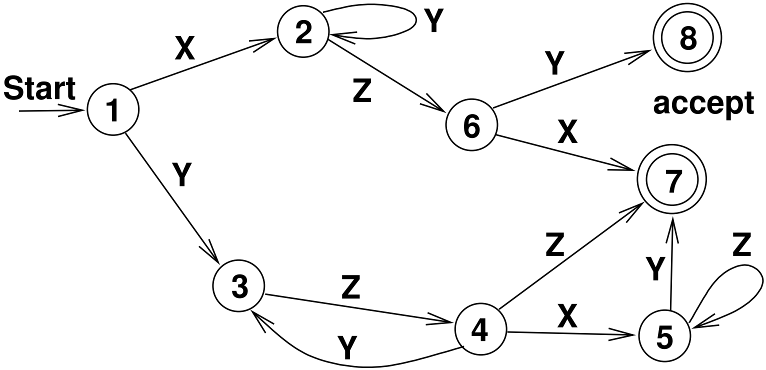
Question: How do we execute Dehn's algorithm efficiently?


Start
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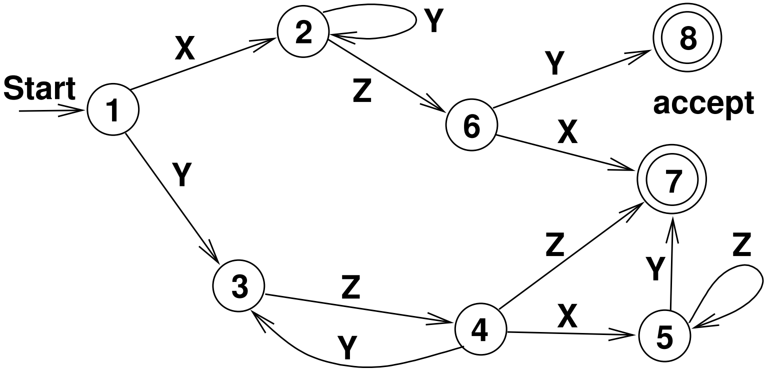

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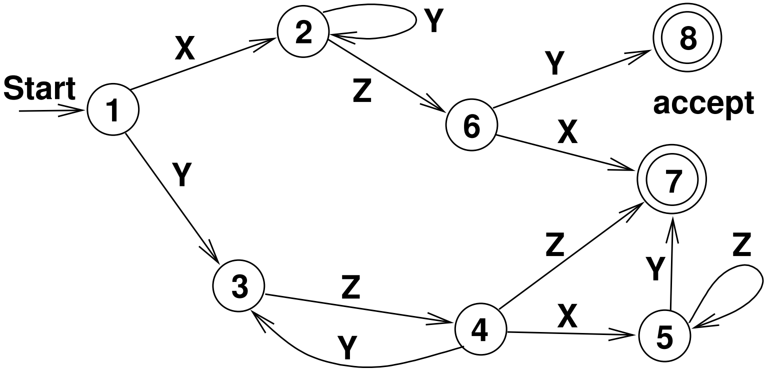

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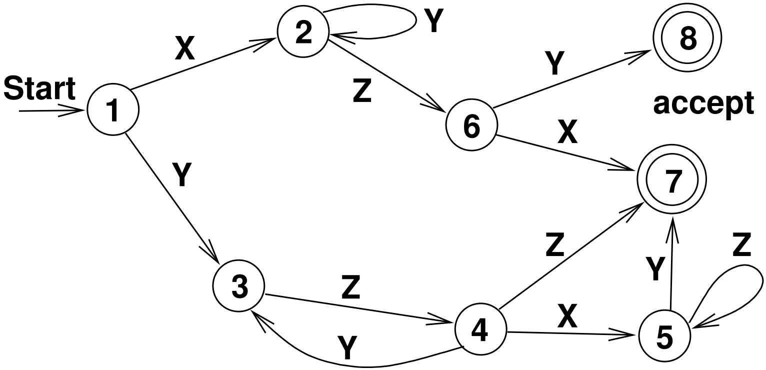

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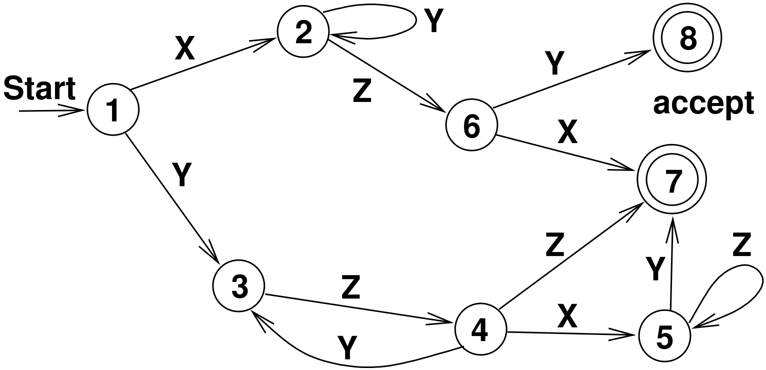

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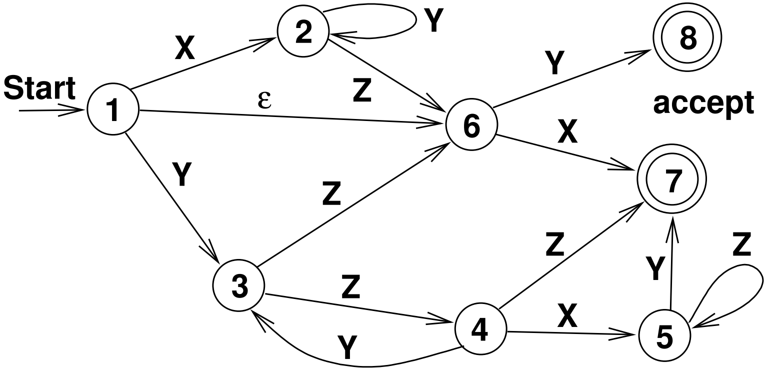
This is a regular language: $XY^*Z(Y + X) + YZ(YZ)^*(Z + XZ^*Y)$.



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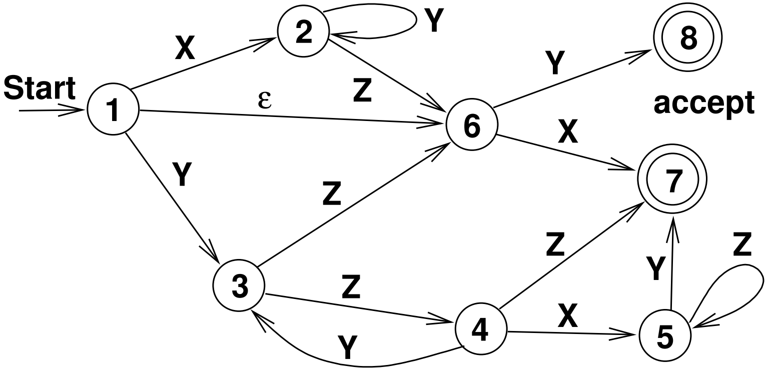
- Allow empty (or ε) transitions.



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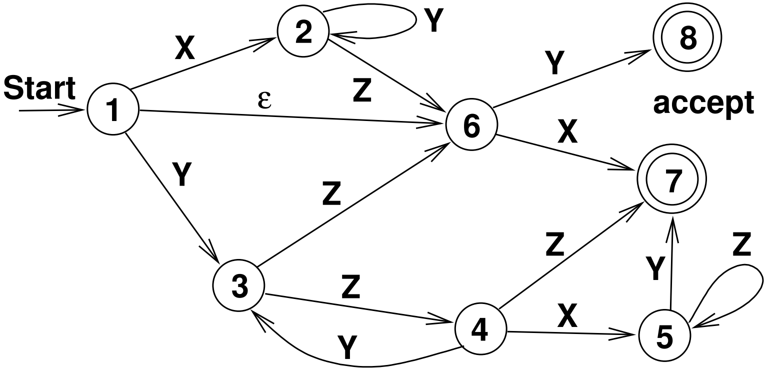

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However: The classes of languages of deterministic and non-deterministic finite state automata are the same.

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$\Longrightarrow$ Very fast algorithm to recognise rewrite rules that apply.
$\Longrightarrow$ Crucial step for Dehn's algorithm.

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Define $p : \hat{X}^* \times \hat{X}^* \to ((\hat{X} \cup \{\$\}) \times (\hat{X} \cup \{\$\}))^*$ by padding the shorter word at the end with \$ symbols:

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Definition (Word acceptor)

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Let G be a group that is generated as a monoid by the set $\hat{X}$. Then G is automatic w.r.t. $\hat{X}$, if there exist FSA W and M_x for $x \in \hat{X} \cup \{\varepsilon\}$, s.th.:

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Definition (Shortlex automatic structure)

If W accepts precisely the shortlex minimal words of $\hat{X}^*$ for the elements of G, then $(W, \{M_x\})$ is a shortlex automatic structure.

Definition (Word differences)

Let $v, w \in \hat{X}^*$ and let v_i be the prefix of v of length i. The word differences of v and w are $D(v, w) := \{v_i^{-1} w_i \mid i \in \mathbb{N}\} \subseteq G$.

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Theorem

Let $(W, \{M_x\})$ be an automatic structure. The set

$$D := \bigcup_{(v,w) \text{ accepted by some } M_x} D(v, w)$$

is finite.

Idea of shortlex automatic structure computation

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`http://tinyurl.com/MNGAPsess/GAP_FP_9.g`

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Thus:

Automatic groups are a large class of groups with solvable word problem.

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