Classifying the non-synchronizing primitive permutation groups

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Introduction

I want to talk about a classification problem for finite primitive permutation groups, which arose from the theory of deterministic finite synchronizing automata.

This permutation group problem turns out to be of interest both theoretically and computationally, and has strong connections to problems in combinatorics, finite geometry, and computation.

Non-synchronizing graphs

Let Γ be a (finite, simple) graph.

- We say that Γ is *non-synchronizing* if Γ is non-null, non-complete, and its chromatic number $\chi(\Gamma)$ equals its clique number $\omega(\Gamma)$.

  (The *chromatic number* of Γ is the smallest number of colours required to make a *proper vertex-colouring* of Γ, that is, a colouring of the vertices so that adjacent vertices have different colours, and the *clique number* of Γ is the size of a largest set of pairwise adjacent vertices.)

Non-synchronizing groups

Let G be a permutation group on a finite set Ω.

- We say that G is *non-synchronizing* if there is a non-synchronizing graph Γ with vertex-set Ω, such that $G \leq \mathrm{Aut}(\Gamma)$.
- It is easy to see that if $|\Omega| > 2$ and G is intransitive or imprimitive, then G is non-synchronizing.
- Now if G is primitive and non-synchronizing, acting as a group of automorphisms of a non-synchronizing graph Γ, then the edge-set of Γ is a union of orbitals of G, and $|\Omega| = \omega(\Gamma)\alpha(\Gamma)$, where $\alpha(\Gamma)$ is the size of a largest set of pairwise non-adjacent vertices of Γ.

GRAPE machinery

The GRAPE package for GAP has for a long time contained powerful machinery for clique finding and classification.

The problem of classifying the non-synchronizing primitive permutation groups motivated me to add much functionality to GRAPE for proper vertex-colouring, and for this, to develop proper vertex-colouring algorithms that exploit the automorphism group of the graph being coloured.

In GRAPE $\geq$ 4.8, where `gamma` is a (simple) GRAPE graph and `k` is a non-negative integer, the function call

```
VertexColouring(gamma, k)
```

returns a proper vertex-colouring of `gamma` using at most `k` colours if such a colouring exists; otherwise, the value `fail` is returned.

The classification so far

Over the past two years or so, I have used:

- the library of primitive permutation groups in GAP,
- the clique and proper vertex-colouring functionality in GRAPE,
- some new theory (thanks to Peter Cameron),
- and some ad hoc computations

to obtain a complete classification of the non-synchronizing primitive permutation groups of degree at most 314.

The GQ(3, 9) has a resolution

Degree 280 was particularly interesting, and I describe one example.

- One primitive group of degree 280 is $U \cong U_4(3).D_8$ in its action on the set L of lines of the (unique) generalised quadrangle GQ(3, 9). To show that U is non-synchronizing, I proved that this GQ(3, 9) has a resolution, that is, a partition of L into 10 parallel classes of 28 lines, a parallel class being a set of lines partitioning the set of 112 points.

- Now the line graph Δ of the GQ(3, 9) has a clique of size 10 (given by 10 lines through a point) and a proper vertex-colouring using just 10 colours (where the colour classes are the parallel classes of a resolution).

- Hence, $10 = \chi(\Delta) = \omega(\Delta)$, so Δ is non-synchronizing.

- Thus $U = \mathrm{Aut}(\Delta)$ is non-synchronizing, and so are all the primitive subgroups of U of degree 280.

Possible future directions

Here are some possible future directions to pursue.

- Adapt my clique and colouring algorithms to be able to use distributed or parallel computation.
- Implement (certain parts) of the GRAPE clique and colouring algorithms in C for efficiency and better integration with *nauty/traces* and the available heuristic proper vertex-colouring programs.
- Push further the classification of the primitive non-synchronizing (and also the broader class of the primitive non-separating) permutation groups.